Distributed Graph Coloring and Related Problems

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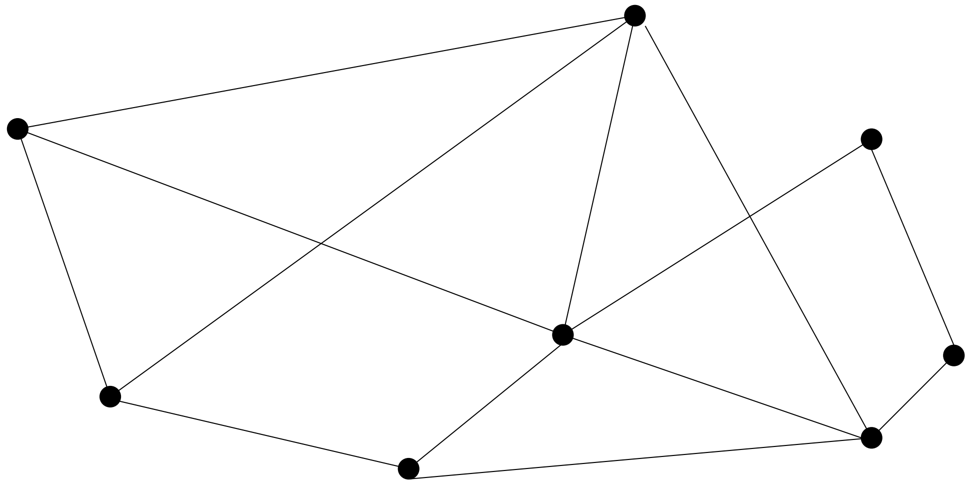
joint w/ L. Barenboim
(PODC'08, STOC'09, PODC'10,
PODC'11, J. ACM'11)

and w/ L. Barenboim, S. Pettie and
J. Schneider (FOCS'12)

The Model



- Unweighted undirected graph $G = (V, E)$.
- Vertices host processors.
- Processors communicate over edges of G.
- Communication is synchronous, i.e., occurs in *discrete* rounds.

- Running time $=$ # rounds.

- All vertices wake up simultaneously.

- Vertices have unique Ids
  from $\{1, 2, \ldots, n\} = [n]$.

- Arbitrarily large messages are allowed,
  though short (of size $O(\log n)$)
  are preferred.

Coloring

- $\Delta = \Delta(G)$ - maximum degree of a vertex in G.

- $\varphi : V \to [k]$ is a k-*vertex-coloring* if $\forall e = (u,w) \in E, \quad \varphi(u) \neq \varphi(w)$.

- $\psi : E \to [t]$ is a t-*edge-coloring* if $\forall e, e'$ s.t. $e \cap e' \neq \emptyset, \psi(e) \neq \psi(e')$.

- In distributed setting, typically $k \geq \Delta + 1, \ t \geq 2\Delta - 1$.

- MIS U:
  (1) $\forall v, w \in U, \quad (v,w) \notin E$.
  (2) $\forall v \notin U, \quad \exists u \in U$ s.t. $(u,v) \in E$.

- MM M:
  (1) $\forall e, e' \in M, e \cap e' = \emptyset$.
  (2) $\forall e' \notin M, \exists e \in M$ s.t. $e \cap e' = \emptyset$.

- $(\Delta + 1)$-coloring in $O(n)$ rounds is easy.

  Color vertices one-by-one:
  For each new vertex v there are
  $\leq \Delta$ forbidden colors.

  Hence there is always an available color
  for v in $[\Delta + 1]$.

- MIS in $O(n)$ rounds is easy too.

  Initialize $U \leftarrow \emptyset$;
  Treat vertices one-by-one:
  For each new vertex v do:
  if $\Gamma(v) \cap U = \emptyset$ then
      v joins U;

- $(2\Delta - 1)$-edge-coloring reduces to
  $(\Delta + 1)$-vertex-coloring,
  MM and $(\Delta + 1)$-vertex-coloring
  reduce to MIS.

Elementary Color Reduction Technique

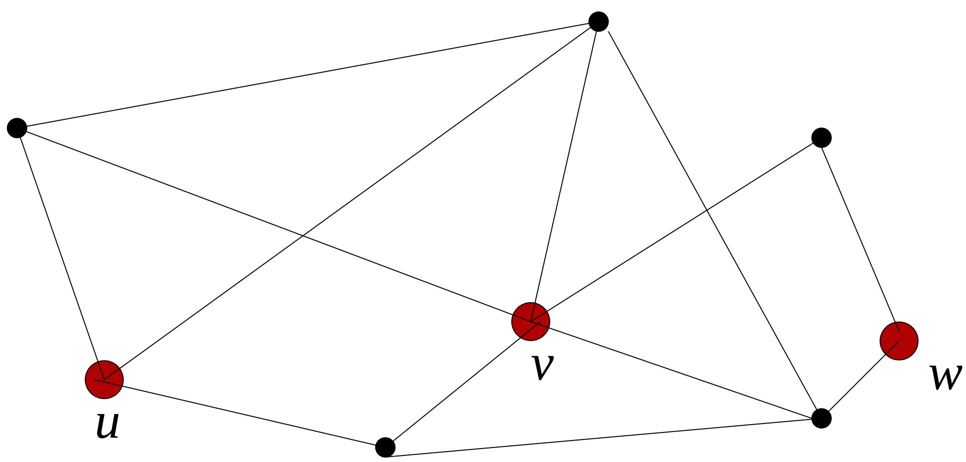
Given an α-coloring, $\alpha > \Delta + 1$,
eliminate one color class in each round.

Vertices of color α form an independent set.

Each of them recolors itself into an available color from $[\Delta + 1]$.

So in $\alpha - (\Delta + 1)$ rounds we get
a $(\Delta + 1)$-coloring.

Contunue with it for $\Delta + 1$ more rounds
to get an MIS.



Kuhn-Wattenhofer's (KW) Color Reduction Technique

$(\Delta + 1)$-coloring in $O(\Delta \log \frac{\alpha}{\Delta+1}) + \log^* n$ time.
[Kuhn,Wattenhofer (PODC'06)]

- Given an α-coloring,
  $\alpha = c \cdot (\Delta + 1)$,
  c is a large integer power of 2.

- $\forall i \in [c]$, let

$$U_i = \{v \mid (i-1) \cdot (\Delta + 1) + 1 \leq \varphi(v) \leq i \cdot (\Delta + 1)\}.$$

- Pair subgraphs $G(U_1)$ with $G(U_2)$,
  $G(U_3)$ with $G(U_4)$,...,
  $G(U_{c-1})$ with $G(U_c)$.

Consider $G(U_1 \cup U_2)$.
It is $2 \cdot (\Delta + 1)$-colored by φ.

- Reduce the $2(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ to get a $(\Delta + 1)$-coloring of $G(U_1 \cup U_2)$ in $2(\Delta + 1) - (\Delta + 1) = \Delta + 1$ rounds.

  In parallel, reduce the colorings of $G(U_3 \cup U_4), G(U_5 \cup U_6), \ldots$
  In $\Delta + 1$ rounds we get $\frac{1}{2}\alpha$-coloring of G.

- Keep halving the #colors by phases that last $\Delta + 1$ rounds each.

  In $\log \frac{\alpha}{\Delta+1}$ phases
  (i.e., in $O(\Delta \cdot \log \alpha / \Delta)$ time)
  we get $(\Delta + 1)$-coloring.

- [Linial (FOCS'87)]:
  $O(\Delta^2)$-coloring in $\log^* n$ time.

  In conjunction with the KW color reduction we get $O(\Delta \log \Delta) + \log^* n$ time for $(\Delta + 1)$-coloring.

- *Locally-iterative* means: in every round every vertex recolors itself based only on colors of its neighbors.

  [Szegedy,Vishwanathan (STOC'92)]:
  Any *locally-iterative* $(\Delta + 1)$-coloring requires $\Omega(\Delta \log \Delta)$ time.

  The $(\Delta + 1)$-coloring algorithms of Linial and of Kuhn and Wattenhofer can be cast as locally-iterative.

  So the KW is an optimal locally-iterative $(\Delta + 1)$-coloring algorithm.

Distributed Coloring - Known Randomized Results

- $(\Delta + 1)$-coloring, MIS and MM in $O(\log n)$ time.
  [Luby (STOC'85)],
  [Alon,Babai,Itai (J.Alg.'86)],
  Israeli,Itai (IPL'86)].

  $(\Delta + 1)$-coloring in $O(\log \Delta + \sqrt{\log n})$ time.
  [Schneider,Wattenhofer (PODC'10)].

- $O(\Delta)$-coloring in $O(\sqrt{\log n})$ time
  [Kothapalli,Scheideler,Onus,
  Schindelhauer (IPDPS'06)].

- $O(\Delta + \log n)$-coloring in $O(\log \log n)$ time, and $O(\Delta \log^{(c)} n + \log^{1+1/c} n)$-coloring in $O(f(c)) = O(1)$ time.
  [Schneider,Wattenhofer (PODC'10)].

New Randomized Algorithms

[Barenboim, E., Pettie, Schneider (FOCS'12)]

- MM in $O(\log\Delta + \log^4 \log n)$ time.

- $(\Delta + 1)$-coloring in
  $O(\log\Delta) + exp\{O(\sqrt{\log\log n})\}$ time.

- $O(\Delta)$-coloring in $exp\{O(\sqrt{\log\log n})\}$ time.

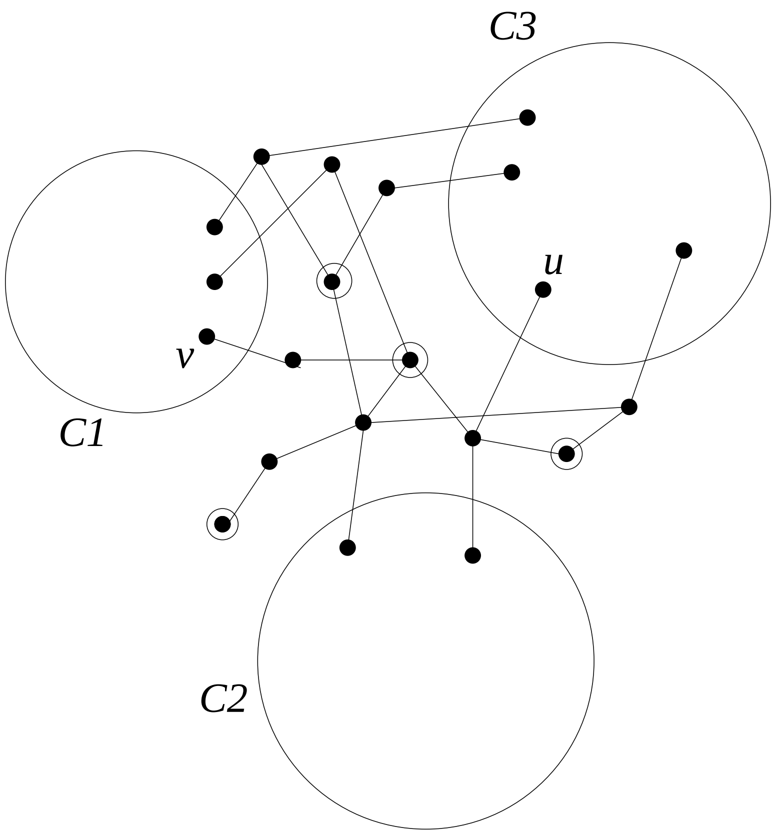
- $\Delta^{1+\eta}$-coloring in $O(\log^2 \log n)$ time.

- $\Delta^{1+\eta}$-edge-coloring in $O(\log\log n)$ time.

- MIS in $O(\log^2\Delta) + exp\{O(\sqrt{\log\log n})\}$ time.

Basic Approach in BEPS's algorithms

- Do (roughly) $O(\log \Delta)$ "Luby" steps to break the graph into disconnected components of size $s \leq \text{polylog}(n)$.



$|C1|, |C2|, |C3| \leq s$.

- Use the state-of-the-art *deterministic* MIS algorithm for each component.

  It completes the MIS within additional $exp\{O(\sqrt{\log s})\} \leq exp\{O(\sqrt{\log\log n})\}$ time.

  Using randomized subroutine within components fails because the failure probability is $1/\text{poly}(s) \approx 1/\text{polylog}(n)$.

- Works similarly for $(\Delta + 1)$-coloring and MM problems.

  For MM the second step requires just $O(\log^4 s) = O(\log^4 \log n)$ time.

- Improved deterministic algorithms give rise to improved randomized ones!

Lower Bounds vs. Upper Bounds

- $f(\Delta)$-coloring requires $\frac{1}{2}\log^* n$ time.

  [Linial (FOCS'87)]

  The upper bound (BEPS)
  for $(\Delta + 1)$-coloring is
  $O(\log \Delta) + \exp\{O(\sqrt{\log\log n})\}$.

  Huge gap!

- Coloring Δ-regular trees in $o(\sqrt{\Delta})$ colors requires $\omega(\log_\Delta n)$ time.

  [Linial (FOCS'87)]

  One can color unoriented forests in
  Δ^ϵ colors within $O(\log_\Delta n)$ time,
  for an arbitrarily small $\epsilon > 0$.

  [Barenboim,E. (PODC'08)] (tight).

- $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time is required for MIS and MM.

  [Kuhn,Moscibroda,Wattenhofer], [(PODC'04), (ArXiv'10)]

  The upper bound (BEPS) for MM is $O(\log \Delta + \log^4 \log n)$.

  Tight for $\log^4 \log n \leq \log \Delta \leq \sqrt{\log n}$.

  For MIS the BEPS's upper bound is $O(\log^2 \Delta) + \exp\{O(\sqrt{\log \log n})\}$.

Known Deterministic Results

- $(\Delta+1)$-coloring and MIS in $O(\Delta^2 + \log^* n)$ time, and in $O(\Delta \log n)$ time.
  [Goldberg,Plotkin,Shannon'87]
  (based on [Cole,Vishkin'86])

- $O(\Delta^2)$-coloring in $\log^* n + O(1)$ time.
  [Linial'87]

  Asked: can one get much fewer than Δ^2 colors in time polylogarithmic in n?

- $(\Delta+1)$-coloring and MIS in $2^{O(\sqrt{\log n})}$ time. (Large messages)
  [Panconesi,Srinivasan'92], based on [Awerbuch,Goldberg,Luby,Plotkin'89]

- MM in $O(\log^4 n)$ time.
  [Hanckowiak,Karonski,Panconesi'99]

- $O(\Delta \cdot \log n)$-edge-coloring in $O(\log^4 n)$ time.

  [Czygrinow,Hanckowiak,Karonski (ESA'01)]

New Deterministic Results

- $(\Delta + 1)$-coloring and MIS in $O(\Delta) + \frac{1}{2}\log^* n$ time.

  [Barenboim,E. (ArXiv'08,STOC'09)], [Kuhn (SPAA'09)]

  Breaks the Szegedy-Vishwanathan's $\Omega(\Delta \log \Delta)$ barrier.

  Major Open Problem:
  The lower bound is only $\frac{1}{2} \cdot \log^* n$ ([Linial'87]),
  while the upper bound is $O(\Delta) + \frac{1}{2}\log^* n$.

- (1) $\Delta^{1+\eta}$-coloring in $O(\log \Delta \cdot \log n)$ time, for any $\eta > 0$.
  (2) $O(\Delta)$-coloring in $O(\Delta^{\epsilon} \cdot \log n)$ time, for any $\epsilon > 0$.

  [Barenboim,E. (PODC'10,J.ACM'11)]

  Answers Linial's question in the affirmative.

  (In polylogarithmic time one can get $\Delta \cdot 2^{O(\log \Delta / \log \log \Delta)}$-coloring.)

- (1) $\Delta^{1+\eta}$-edge-coloring in $O(\log \Delta + \log^* n)$ time, for any $\eta > 0$.

  (2) $O(\Delta)$-edge-coloring in $O(\Delta^{\epsilon} + \log^* n)$ time, for any $\epsilon > 0$.

  [Barenboim,E. (PODC'11)]

Special Families of Graphs: Bounded Arboricity

Planar graphs:
MIS and other problems can be solved in deterministic $O(\log n)$ time.
[Goldberg,Plotkin,Shannon'87]

Arboricity $a = a(G)$, $G = (V, E)$

$$a = \max_{U \subseteq V, |U| \geq 2} \{\lceil \frac{|E(U)|}{|U| - 1} \rceil\}$$

Forests have arboricity 1.

Planar graphs have arboricity ≤ 3.

Graphs of bounded genus or treewidth have bounded arboricity.

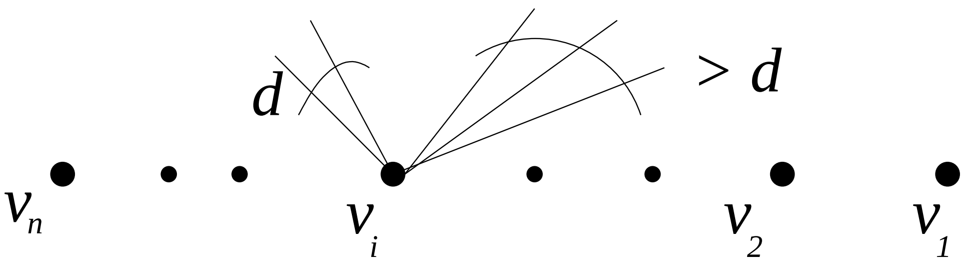
Graphs that exclude any fixed minor have bounded arboricity.

Arboricity (Continued)

Nash-Williams's Thm'61:
The arboricity $a = a(G)$ is the minimum number of edge-disjoint forests required to cover G.

arboricity $\approx$ degeneracy.

$degen(G) = d$ is the minimum number s.t. $V = V(G)$ can be ordered $v_1, v_2, \ldots, v_n$, and each v_i has $\leq d$ edges $(v_i, v_j), i < j$.



Given the ordering it is easy to $(d+1)$-color the graph.

New Results for Graphs with Bounded Arboricity

- (1) $(2+\epsilon)a$-coloring in $O(a \cdot \log n)$ deterministic time.
  (2) $O(a^2)$-coloring in $O(\log n)$ deterministic time.
  (3) $\forall q,\ O(q \cdot a^2)$-coloring in $O(\log_q n)$ deterministic time.

  [Barenboim,E.'(PODC'08)]

- $\forall q,\ \sqrt{\log n} \leq \log q \leq \frac{\log n}{\log \log n}$,

  $O(q \cdot a)$-coloring in $O(\log_q n)$ randomized time.

  [Kothapalli,Pemmaraju(PODC'11)]

- A lower bound of $\Omega(\log_q n)$ for $(q \cdot a)$-coloring.

  [Barenboim,E.'08], based on [Linial'87].

Bounded Arboricity (Continued)

- MIS and $(\Delta + 1)$-coloring
  in $O\left(\frac{\log n}{\log\log n}\right)$ deterministic time,
  for $a \leq \log^{1/2-\epsilon} n$.

- MM and $(2\Delta - 1)$-edge-coloring
  in $O\left(\frac{\log n}{\log\log n}\right)$ deterministic time,
  for $a \leq \log^{1-\epsilon} n$.

- For $a \leq \text{polylog}(n)$,
  MIS, MM, $(\Delta + 1)$-coloring
  and $(2\Delta - 1)$-edge-coloring
  can all be solved
  in deterministic $\text{polylog}(n)$ time.

  [Barenboim,E.'08]

- $(2+\epsilon)^k \cdot a$-coloring
  in $a^{O(1/k)} \log n$ deterministic time,
  $\forall k = 1, 2, .., \forall \epsilon > 0$.

  Means: $O(a)$-coloring in $a^\epsilon \cdot \log n$
  deterministic time, $(\forall \epsilon > 0)$.

  Also, $a^{1+\eta}$-coloring in $O(\log a \cdot \log n)$
  deterministic time. $(\forall \eta > 0)$

  Implies: $\Delta^{1+\eta}$-coloring in $O(\log \Delta \cdot \log n)$
  deterministic time. $(\forall \eta > 0)$.

  Also, if $a \le \Delta^{1-\epsilon}$ we get
  $(\Delta + 1)$-coloring
  in deterministic polylog(n) time.

  [Barenboim,E. (PODC'10,J.ACM'11)]

Bounded Arboricity:
New Randomized Results

- MM: $O(\log a + \sqrt{\log n})$. (BEPS)
  Lower bound: $\Omega(\sqrt{\log n})$,
  even for unoriented trees.
  BEPS,
  based on [Kuhn,Moscibroda,Wattenhofer'04]

  Tight for $1 \leq a \leq \exp\{\sqrt{\log n}\}$.
  Open for larger values of a.

- MIS: $O(\log^2 a + \log^{2/3} n)$. (BEPS).
  For trees $O(\sqrt{\log n \log \log n})$ (BEPS),
  refining $O(\sqrt{\log n} \log \log n)$ bound due to
  [Lenzen,Wattenhofer (PODC'11)].

  No lower bound of $\sqrt{\log n}$
  for MIS in unoriented trees is known!

Graphs with Small Arboricity: Basic Technique

Observation 1:
In an n-vertex graph $G = (V, E)$
with $a(G) = a$, there exists
a *constant fraction* of vertices (subset H) s.t.
$\forall v \in H, \quad deg(v) \leq 3 \cdot a$.

It extends the notion of *degeneracy*:
a graph of degeneracy d
must contain at least *one* vertex v
with $deg(v) \leq d$.

Observation 2: $a(G(V \setminus H)) \leq a(G)$

$$\Downarrow$$

We can extract such sets H many times,
and get an H-*partition* of G.

The Peeling Process: H-decomposition

Iteratively remove low-degree sets $H_1, H_2, \ldots$.

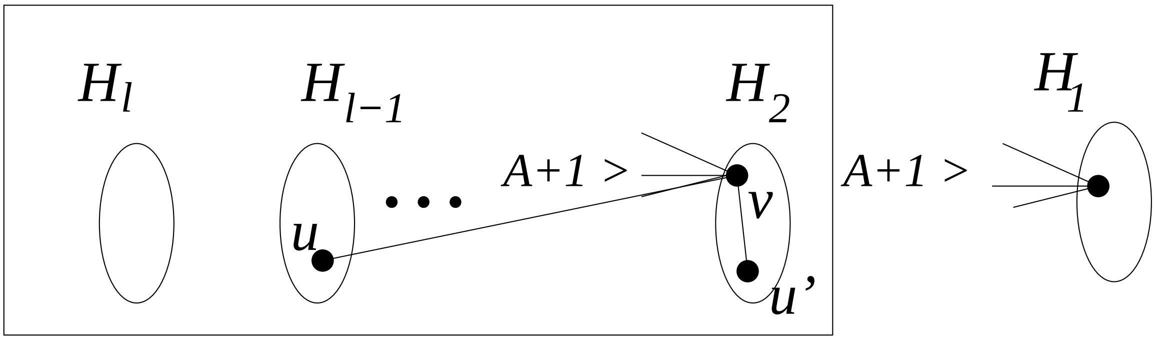
For some ℓ, all vertices v in $H_\ell = V \setminus \bigcup_{i=1}^{\ell-1} H_i$ have $deg(v, H_\ell) \leq 3 \cdot a$.

H_ℓ is the last set in the H-decomposition.

ℓ - the number of H-sets.

On each step at least a constant fraction of vertices is eliminated.

$\ell = O(\log n)$.



$A = 3 \cdot a.$

$V = \bigcup_{i=1}^{\ell} H_i, \quad H_i \cap H_j = \emptyset, \ \forall i \neq j$

$\forall i \in [\ell], \ \forall v \in H_i, \ deg(v, \bigcup_{j=i}^{\ell} H_j) \leq A.$

In particular, $deg(v, H_i) \leq deg(v, \bigcup_{j=i}^{\ell} H_j) \leq A.$

The H-decomposition can be computed
in $O(\ell) = O(\log n)$ time.
(One round for each H_i.)

[Zhou,Nishizeki'95],
[Barenboim,E.'08]

Coloring Using H-Decomposition

- Compute an H-decomposition $H_1, H_2, \ldots, H_\ell$ in $O(\ell) = O(\log n)$ time.

- In parallel, in each H_i compute an $(A+1)$-coloring φ in $O(A + \log^* n)$ time. $(\Delta(H_i) \leq A)$

- Recolor to obtain an $(A+1)$-coloring ψ of the entire original graph G.

  On this step we spend $O(A \cdot \ell) = O(a \cdot \log n)$ time.

Recoloring (Producing ψ)

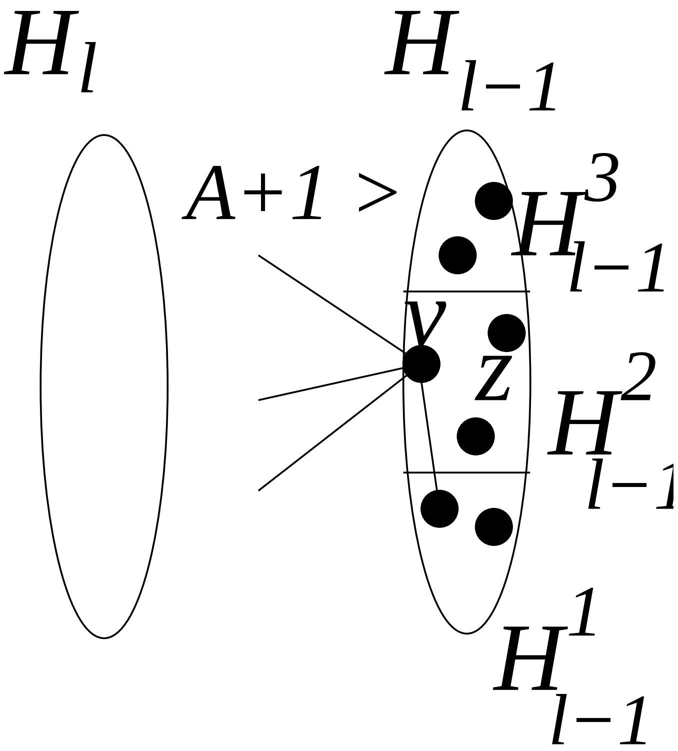
Spend $(A+1)$ rounds on each set H_i.

Start with H_ℓ.
Each $v \in H_\ell$ sets $\psi(v) \leftarrow \varphi(v)$.

Proceed to $H_{\ell-1}$.
$\forall r \in [A+1]$,
$H^r_{\ell-1} = \{v \in H_{\ell-1} \mid \varphi(v) = r\}$.

Recolor one φ-color class at a time.
(Each φ-color class is an independent set.)

Suppose for some $r \in [A]$ that
$H^1_{\ell-1} \cup \ldots \cup H^r_{\ell-1}$ are already recolored.

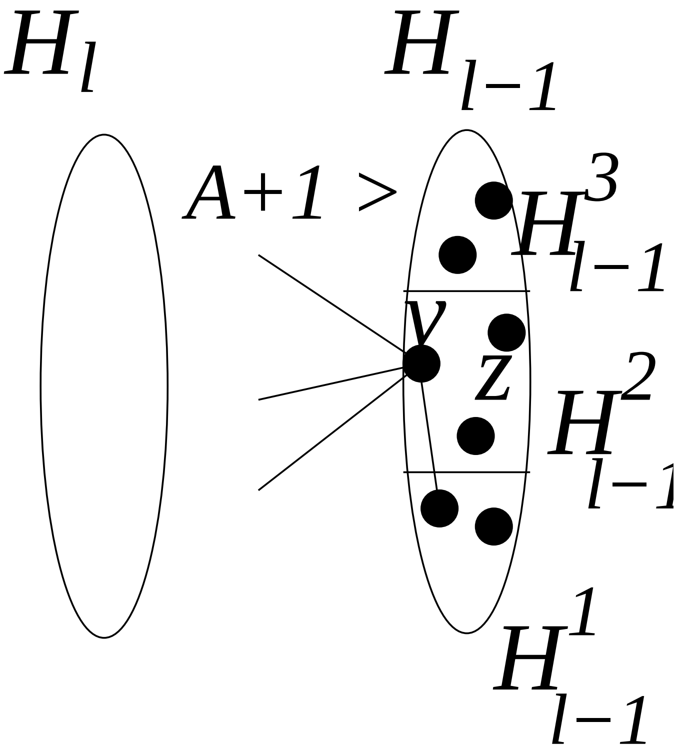


Consider $v \in H_{\ell-1}^{r+1}$.

v has $\leq A$ neighbors in $H_\ell \cup H_{\ell-1}$.

$$\Downarrow$$

v has $\leq A$ recolored neighbors.
(Because those are in $H_\ell \cup \bigcup_{j=1}^{r} H_{\ell-1}^{j}$.)



Hence there is a color $c = c(v) \in [A+1]$
s.t. no recolored neighbor u of v
has $\psi(u) = c$.

All vertices $v \in H^{r+1}_{\ell-1}$
compute in parallel $c(v)$ and
set $\psi(v) \leftarrow c(v)$.

Since $H^{r+1}_{\ell-1}$ is an independent set,
the new coloring ψ is legal.

The algorithm:

Recolor $H^1_{\ell-1}$, then $H^2_{\ell-1}, \ldots, H^{A+1}_{\ell-1}$;
then recolor $H^1_{\ell-2}, H^2_{\ell-2}, \ldots, H^{A+1}_{\ell-2}$;

$$\vdots$$

$$H^1_1, H^2_1, \ldots, H^{A+1}_1.$$

There are $A+1$ color classes in each H_i, and ℓ sets H_i.

One round per color class.
Overall $O((A+1)\cdot\ell) = O(a\cdot\log n)$ time.

Thm: $O(a)$-coloring can be computed in $O(a\cdot\log n)$ time.

[Barenboim,E.'08]

It generalizes a 7-coloring algorithm for planar graphs.
[Goldberg,Plotkin,Shannon'87]

Basic Building Blocks for Further Progress

- **Defective coloring:**
  For $(\Delta + 1)$-coloring in $O(\Delta) + \log^* n$ time.
  [Barenboim,E. (STOC'09)],
  [Kuhn (SPAA'09)]

  Enables one to bypass the
  Szegedy-Vishwanathan's barrier
  of $\Omega(\Delta \log \Delta)$ for
  locally-iterative algorithms.

- **Arbdefective coloring:**
  For $\Delta^{1+\eta}$-coloring
  in $O(\log \Delta \cdot \log n)$ deterministic time.
  [Barenboim,E. (PODC'10,J.ACM'11)]

  Answering in the affirmative
  Linial's open question.

$(\Delta+1)$-Coloring in $O(\Delta) + \log^* n$ Time (Defective Coloring)

[Burr,Jacobson'85],[Harary,Jones'86]
[Cowen,Cowen,Woodall'86]

Def: The *defect* of a vertex v wrt coloring φ is the number of neighbors $u \in \Gamma(v)$ with $\varphi(u) = \varphi(v)$.

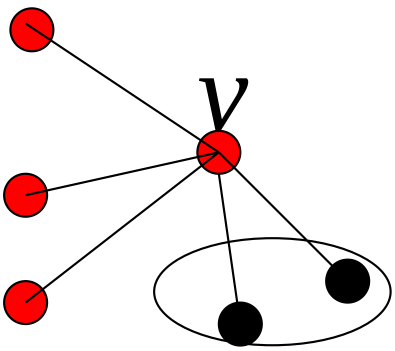
Def: The *defect* d of a k-coloring φ is the maximum defect of a vertex wrt φ. φ is called a d-*defective* k-*coloring*.

Thm: [Lovasz'66]
$\forall G,\ \forall p$ there exists a $\lfloor \Delta/p \rfloor$-defective p-coloring of G.

Proof of Lovasz's Thm

φ - an arbitrary p-coloring.
(Not necessarily legal or Δ/p-defective.)

while $\exists v$ with defect$(v) > \Delta/p$ *do*
{
$\varphi(v) \leftarrow$ the color used by
min. #neighbors of v;
}



Delta = 5,
p = 2,
there exists
a color used
by 2 < 5/2
neighbors

ME_i - the total #monochromatic edges before iteration i starts.

$$\mathrm{ME}_{i+1} = \mathrm{ME}_i - \mathrm{defect}(v) + \lfloor \frac{\Delta}{p} \rfloor < \mathrm{ME}_i.$$

But $0 \leq ME_i \leq |E|$, and so within a finite number of iterations this process terminates.

Distributed Counterparts of Lovasz's Theorem

Thm: [Barenboim,E. (STOC'09)]
$\forall G,\ \forall p\ \lfloor \Delta/p \rfloor$-defective $O(p^2)$-coloring of G can be computed in $O(\Delta^{\epsilon}) + \frac{1}{2}\log^* n$ time, $\forall \epsilon > 0$.

Thm: [Kuhn (SPAA'09)]
$\forall G,\ \forall p\ \lfloor \Delta/p \rfloor$-defective $O(p^2)$-coloring of G can be computed in $O(\log^* \Delta) + \frac{1}{2}\log^* n$ time.

Open: can one efficiently achieve a linear (in Δ) product of defect and #colors?

Partial answer: for *edge*-coloring it is possible.
Also, for vertex-coloring of graphs with bounded independence.
[Barenboim,E. (PODC'11)]

$(\Delta + 1)$-Coloring Algorithm

- Compute $O\left(\frac{\Delta}{\log \Delta}\right)$-defective $\log^2 \Delta$-coloring of G in $o(\Delta) + O(\log^* n)$ time.
  ($p = \log \Delta$)

- Each color class induces a subgraph with maximum degree $\Delta' = O\left(\frac{\Delta}{\log \Delta}\right)$.

  Subgraphs are vertex-disjoint.

- In parallel, compute $(\Delta' + 1)$-coloring in each of the $\log^2 \Delta$ subgraphs in $O(\Delta' \log \Delta' + \log^* n) = O(\Delta + \log^* n)$ time, using KW algorithm.

- Overall we get $O((\Delta' + 1) \log^2 \Delta) = O(\Delta \log \Delta)$-coloring φ of the entire original graph.

  (Using distinct palettes.)

- Invoke KW iterative procedure.

  Given α-coloring it returns
  $(\Delta + 1)$-coloring in $O(\Delta \cdot \log \frac{\alpha}{\Delta})$ time.

  For $\alpha = \Delta \log \Delta$,
  the time is $O(\Delta \log \log \Delta)$.

  Overall running time is
  $O((\Delta + 1) \cdot \log \log \Delta + \log^* n) + o(\Delta)$.

This is a self-improving scheme!

Now we have $(\Delta + 1)$-coloring algorithm that runs in $O(\Delta \log \log \Delta + \log^* n)$ time.

- Compute $O\left(\frac{\Delta}{\log \log \Delta}\right)$-defective
  $(\log \log \Delta)^2$-coloring
  in $o(\Delta) + O(\log^* n)$ time.

- $\Delta' = \frac{\Delta}{\log \log \Delta}$.

Compute $(\Delta' + 1)$-coloring of each subgraph in $O(\Delta' \log\log \Delta' + \log^* n) = O(\Delta + \log^* n)$ time.

- Combine these colorings into an $O(\Delta \log\log \Delta)$-coloring of G (in zero time).

- Reduce the $O(\Delta \cdot \log\log \Delta)$-coloring via KW iterative procedure into a $(\Delta + 1)$-coloring within $O(\Delta \cdot \log^{(3)} \Delta + \log^* n)$ additional time.

  Overall we get $(\Delta + 1)$-coloring in $O(\Delta \cdot \log^{(3)} \Delta + \log^* n)$ time.

$$\Downarrow$$

Repeating this argument $\log^* \Delta$ times we get $(\Delta + 1)$-coloring in $O(\Delta + \log^* n)$ time.

A tradeoff (an application)

$\forall t$, $O(\Delta \cdot t)$-coloring in $O(\Delta/t + \log^* n)$ time. (Interpolates between Linial's $O(\Delta^2)$-coloring in $\log^* n$ time, and our $(\Delta + 1)$-coloring in $O(\Delta + \log^* n)$ time.)

- Compute (Δ/t)-defective $O(t^2)$-coloring in $O(\log^* n)$ time.

- We get $O(t^2)$ vertex-disjoint subgraphs, each with $\Delta' \leq \Delta/t$.

  Compute $(\Delta' + 1)$-coloring of each, in parallel, in $O(\Delta' + \log^* n) = O(\Delta/t + \log^* n)$, using the last result for $(\Delta' + 1)$-coloring.

- Combine the colorings in zero time to get $O(t^2 \cdot \Delta') = O(\Delta \cdot t)$-coloring, in total $O(\Delta/t + \log^* n)$ time.

Open Questions

1. A $(\Delta + 1)$-coloring or an MIS in deterministic polylogarithmic time?

   Or at least $O(\Delta)$-coloring.

   Currently we have $\Delta \cdot 2^{O(\frac{\log \Delta}{\log \log \Delta})}$-coloring.

2. A $\Delta^{2-\epsilon}$-coloring in sublogarithmic time?

3. A $(\Delta + 1)$-coloring in $o(\Delta)$ time? Or a lower bound?

   Currently we have $O(\Delta) + \frac{1}{2}\log^* n$ time.

4. Δ/p-defective $O(p)$-coloring in deterministic polylogarithmic time? (Known for edge-coloring, and for vertex-coloring of graphs with bounded neighborhood independence.)

5. $(2a+1)$-coloring faster than in $O(a^2 \log n)$ time?
$(2+\eta)\cdot a$-coloring faster than in $O(a \log n)$ time?

We know
$(2+\eta)^{1/\epsilon}a$-coloring in $O(a^\epsilon \cdot \log n)$ time, and $a^{1+\eta}$-coloring in $O(\log a \cdot \log n)$ time.

There is also a *lower bound* of $\Omega\left(\frac{\log n}{\log a}\right)$ for $O(a^2)$-coloring.

So unlike graphs with bounded degree, for graphs of bounded arboricity one cannot hope for sublogarithmic time.

6. MIS or MM in randomized $o(\log n)$ time, for *all* values of Δ (or a)?

7. Randomized MIS in planar graphs in $o(\log^{2/3} n)$ time? Or a lower bound?

- More details can be found in my monograph, joint with Leonid Barenboim, titled "Distributed Graph Coloring", Morgan-Claypool publishing house, Distributed Computing Series, ed. by Nancy Lynch.

  See my web-page www.cs.bgu.ac.il/elkinm.

- Looking for grad. students and/or postdocs to work on this stuff!

Thank you!!